Classical & Quantum Waves | Lecture 12-1

Wave equation — Super important!

→ Describes all wave motion in any system. Universal!

- Recall that general form for any wave motion is:

$$y = \underbrace{f(x-vt)}_{\text{Describes waves moving in } +\hat{x} \text{ direction}} + \underbrace{g(x+vt)}_{\text{" " } -\hat{x} \text{ direction}}$$

→ General solution to "wave eq." in 1-dimension

- Let's derive it. Simplify math with change of variables

$$\rightarrow f(x-vt) \equiv f(u)$$

[consider only f(x-vt) here b/c similar analysis applies to g(x+vt)]

- Write out its derivatives

$$\frac{\partial f}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x}$$

["Chain rule" of calculus for 1st derivative]

$$\frac{\partial^2 f}{\partial x^2} = \frac{d^2 f}{du^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{df}{du}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

[Chain rule for 2nd derivative]

- Since $\cancel{\frac{\partial u}{\partial x} =}$ $u = x - vt$, $\frac{\partial u}{\partial x} = 1$ $\quad \frac{\partial^2 u}{\partial x^2} = 0$

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} = \frac{d^2 f}{du^2}}$$

- Now need to repeat for time derivative $\frac{\partial}{\partial t}$

- Chain rules: $\frac{\partial f}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t}$

$$\frac{\partial^2 f}{\partial t^2} = \frac{d^2 f}{du^2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{df}{du}\left(\frac{\partial^2 u}{\partial t^2}\right)$$

- $u = x - vt \Rightarrow \frac{\partial u}{\partial t} = -v \ , \ \frac{\partial^2 u}{\partial t^2} = 0$

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial t^2} = \frac{d^2 f}{du^2} v^2}$$

- ~~E~~ Plugging in our previous result $\frac{\partial^2 f}{\partial x^2} = \frac{d^2 f}{du^2}$

$$\Rightarrow \underbrace{\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}}$$

2^{nd} derivatives of space & time related by factor of $(\text{velocity})^2$

[this is already the essence of the wave eq.]

- Can apply a similar procedure to $g(x+vt)$. Only difference is that $\frac{\partial u}{\partial t} = +v$, but since we only use $\left(\frac{\partial u}{\partial t}\right)^2$ this doesn't matter.

$$\Rightarrow \frac{\partial^2 g}{\partial t^2} = v^2 \frac{\partial^2 g}{\partial x^2}$$

- Combine these two results:

$$\frac{\partial^2(f+g)}{\partial t^2} = v^2 \frac{\partial^2(f+g)}{\partial x^2}$$ or, since $y = f+g$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$$ This is the <u>1</u>-dimensional wave eq.!

→ Describes all waves in 1D, and the parameter "y" can be replaced with many other things to describe different types of waves (e.g., voltage, pressure, temperature ...)

- As a specific example, take traveling sinusoidal wave discussed previously: $y = A\sin[2\pi(x-vt)/\lambda]$

- Take derivatives: $$\frac{\partial y}{\partial x} = \left(\frac{2\pi}{\lambda}\right) A\cos\left[\frac{2\pi}{\lambda}(x-vt)\right]$$

$$\frac{\partial^2 y}{\partial x^2} = -\left(\frac{2\pi}{\lambda}\right)^2 A\sin\left[\frac{2\pi}{\lambda}(x-vt)\right] \quad [\text{eq. } 5.24]$$

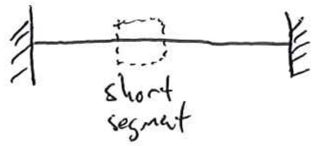
$$\frac{\partial y}{\partial t} = -\frac{2\pi v}{\lambda} A\cos\left[\frac{2\pi}{\lambda}(x-vt)\right]$$

$$\frac{\partial^2 y}{\partial t^2} = -\left(\frac{2\pi v}{\lambda}\right)^2 A\sin\left[\frac{2\pi}{\lambda}(x-vt)\right] \quad [\text{eq. } 5.25]$$

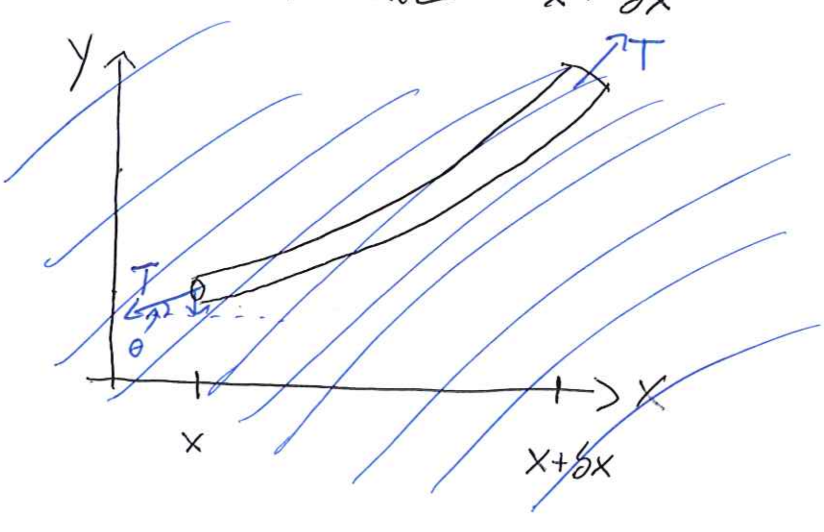
Divide $\frac{[5.25]}{[5.24]}$ $\Rightarrow \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$ ✓

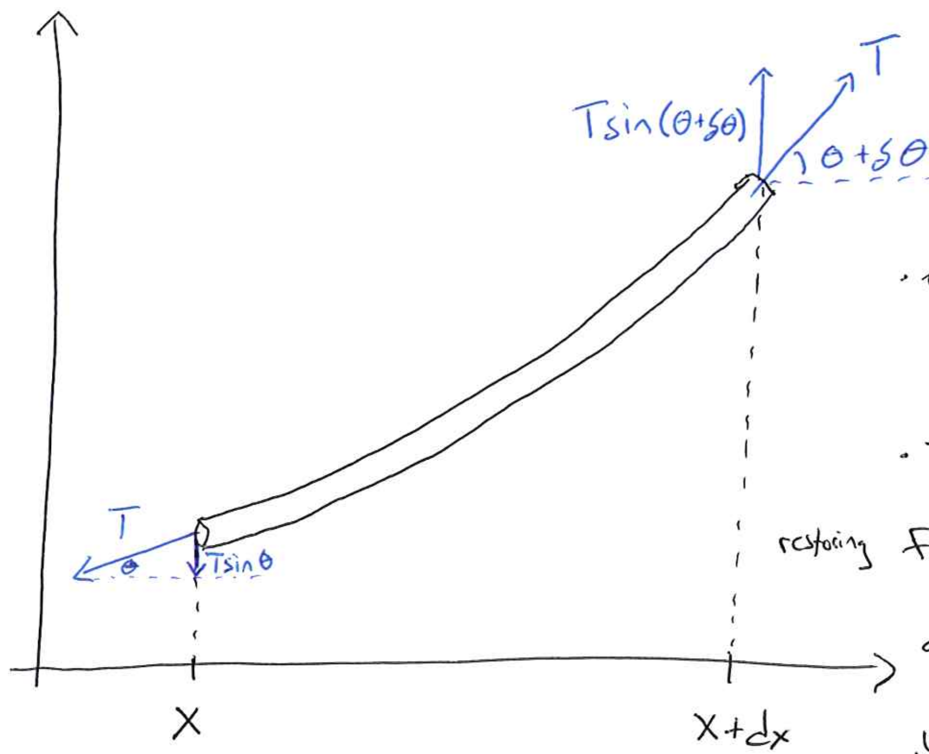
Vibrating String

- Let's derive the eq. of motion for <u>transverse</u> vibrations on a taut string



- Consider a short segment and the forces that act on the string as the wave passes by
- String has mass per unit length μ and is held under tension T
- Wave propagates in x-direction and we consider displacements along y
- For small displacements along y, the tension T can be assumed to be constant (see lecture 10 notes)
- Consider the forces at play in a very small segment between x and $x + \delta x$





- Angle θ at x
  $\theta + \delta\theta$ at $x + \delta\theta$
- This leads to different restoring forces in $\hat{x}$ and $\hat{y}$ directions at x and $x+\delta x$ positions b/c of tension

- First consider the forces in y-direction
- At position x, y-direction force $F_y = T\sin\theta$
- For small angles, $\sin\theta \sim \theta \sim \underline{\tan\theta}$ $\left[\tan\theta = \frac{\sin\theta}{\cos\theta}\right.$ $\cos\theta \sim 1$ for small $\measuredangle$ $\left.\right]$
- We use this approx. b/c $\tan\theta = \frac{dy}{dx}$ → slope of the rope

$$\Rightarrow F_y = T\frac{dy}{dx} \quad \text{at position } x$$

- Similarly, transverse force at $x+\delta x$ is equal to T times slope at that point. [$T\sin(\theta+\delta\theta)$ also]
- What is the slope at $x+\delta x$?

→

(slope at $x+\delta x$) = (slope at x) + (rate of change of slope) $\times \delta x$

[for small θ where slope varies smoothly]

$$\rightarrow \left(\frac{\partial y}{\partial x}\right)_{x+\delta x} = \left(\frac{\partial y}{\partial x}\right)_x + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\delta x$$

$$= \left(\frac{\partial y}{\partial x}\right)_x + \left(\frac{\partial^2 y}{\partial x^2}\right)\delta x$$

- So $F_y(x+\delta x) = T\left[\left(\frac{\partial y}{\partial x}\right)_x + \left(\frac{\partial^2 y}{\partial x^2}\right)\delta x\right] = T\sin(\theta + d\theta)$

force evaluated at $x+\delta x$

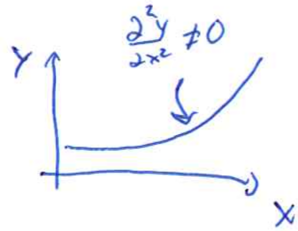
- Consider the total force acting on segment of rope:

$$F_y^{total} = T\sin(\theta+\delta\theta) - T\sin\theta = T\left[\left(\frac{\partial y}{\partial x}\right)_x + \left(\frac{\partial^2 y}{\partial x^2}\right)\delta x - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

minus b/c T in opp. directions

$$\boxed{F_y^{total} = T\left(\frac{\partial^2 y}{\partial x^2}\right)\delta x}$$

$F_y^{total} \neq 0$ only when there is curvature in the rope



- Now consider forces in x-direction
- 2 ends of ~~rope~~ segment experience opposing forces along x
- at position x, $F_x = -T\cos\theta$, at $x+\delta x$ $F_x = T\cos(\theta+\delta\theta)$
- $F_x^{total} = T\cos(\theta+\delta\theta) - T\cos\theta$
- Since θ small, $\cos\theta \sim \cos(\theta+\delta\theta) \sim 1$

$$\Rightarrow \boxed{F_x^{total} = 0}$$

- Returning to the y-direction force. Applying Newton's law:

$F = ma$, mass $= \mu\delta x$, $a = \frac{\partial^2 y}{\partial t^2}$

μ: $\frac{\text{mass}}{\delta x}$

$$\Rightarrow \mu\delta x \frac{\partial^2 y}{\partial t^2} = \underbrace{T\left(\frac{\partial^2 y}{\partial x^2}\right)\delta x}_{F_y^{total}}$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} \quad \Rightarrow \quad \frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad \text{(wave eq.)}$$

We derived wave eq! w/ $v = \sqrt{\frac{T}{\mu}}$

wave velocity set by properties of medium $\longrightarrow$ change tension in guitar string, $\rightarrow$ tunes tone

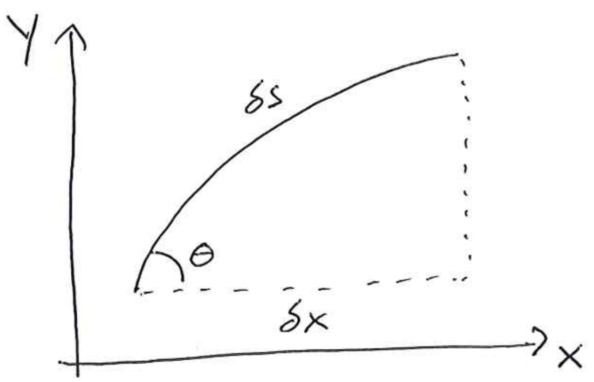
Energy in vibrating string

- Following the same example as above, the kinetic energy is:

$$K = \frac{1}{2} m v^2 = \frac{1}{2} \mu \delta x \left(\frac{\partial y}{\partial t}\right)^2$$

- What about potential energy? Since string is under tension, this gives potential energy equal to the (extension) x (tension)

assume constant



- approx. $\delta s = \dfrac{\delta x}{\cos\theta} = \dfrac{\delta x}{(1-\sin^2\theta)^{1/2}}$

helpful to rewrite this way

- use small ∡ approx

$$\Rightarrow \delta s \approx \frac{\delta x}{(1-\theta^2)^{1/2}}$$

- use expansion $(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + \ldots$

w/ $x = -\theta^2$, $n = -\frac{1}{2}$

$$\Rightarrow \delta s \approx \delta x\left(1 - \tfrac{1}{2}(-\theta^2)\right) = \delta x\left(1 + \frac{\theta^2}{2}\right)$$

- As earlier, we use again the approx. $\theta \sim \tan\theta = \frac{\partial y}{\partial x}$ (for small angles)

$$\Rightarrow \delta s \approx \delta x\left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]$$

- A good approx of potential energy is then

$$U = T\underbrace{(\delta s - \delta x)}_{\text{amount string stretched}} = \frac{1}{2} T \delta x \left(\frac{\partial y}{\partial x}\right)^2$$

Recap:

$$\Rightarrow \quad K = \frac{1}{2}\mu \delta x \left(\frac{\partial y}{\partial t}\right)^2 \qquad U = \frac{1}{2} T \delta x \left(\frac{\partial y}{\partial x}\right)^2$$

- Since these expressions are both ~~per~~ for differential lengths δx, need to integrate expressions over length of interest, say between $x = a$ and $x = b$

$$\Rightarrow E = \frac{1}{2}\int_a^b dx \left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]$$

use $v = \sqrt{\frac{T}{\mu}} \Rightarrow \cancel{B}\; T = \frac{v^2}{\mu}$

$$\Rightarrow \boxed{E = \frac{1}{2}\mu \int_a^b dx \left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right]}$$

Applies to any transverse wave on string

· Example. Energy of a sine wave $y = A\sin(kx - \omega t)$

on length of string b/t x_0 and $x_0 + \lambda$ (λ: wavelength)

· To calculate kinetic energy, need $\frac{\partial y}{\partial t}$

$$\frac{\partial y}{\partial t} = -\omega A \cos(kx - \omega t)$$

$$\Rightarrow K = \frac{1}{2}\mu \delta x \left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\mu \delta x \, \omega^2 A^2 \cos^2(kx - \omega t)$$

$$K_{\substack{\text{total}\\ \text{length}}} = \frac{1}{2}\mu\omega^2 A^2 \underbrace{\int_0^\lambda \cos^2(kx - \omega t)\, dx}$$

let $x_0 = 0$

$$= \frac{\lambda}{2} \quad \text{(with fair amount of math)}$$

$$\Rightarrow K_{total} = \frac{1}{4}\mu\omega^2 A^2 \lambda$$

· Potential energy: $U = \frac{1}{2} v^2 \mu \delta x \left(\frac{\partial y}{\partial x}\right)^2 = \underbrace{\frac{1}{2} v^2 \mu \delta x \, k^2 A^2 \cos^2(kx - \omega t)}$

Following same math steps as above but for $\frac{\partial y}{\partial x}$ instead of $\frac{\partial y}{\partial t}$

· Integrate in similar way to above

$$U_{total} = \frac{1}{4}\mu\omega^2 A^2 \lambda$$

$$\Rightarrow \boxed{E_{total} = K_{tot} + U_{tot} = \frac{1}{2}\mu\omega^2 A^2 \lambda}$$